8.1 Birth-Death Processes

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Number X of people, animals, bacteria etc. in a system.

General Markov Queue - useful population model, called a *birth-death* process, relabel $Q_t \to X_t$.

Markov means births, deaths are independent, exponentially distributed.

General Markov birth-death process allows general state dependent birth rates a_i & death rates s_i for $i \in \{0, 1, 2, ...\}$ provided $s_0 = 0$ so population does not go negative.

$a_0 = 0$ would mean population could die out.

$X(0)$ is initial population size - could be random.

Linear Growth Model

Simple models scale births and deaths with population i

A continuous Markov process is said to be a *linear growth model* if there are constants $\lambda, \mu \geq 0$ such that $a_i = i\lambda$ and $s_i = i\mu$.

i is the population size as well as the state of Markov chain.

Immigration

If we add a constant rate of immigration θ

$$\begin{aligned} a_i &= i\lambda + \theta \\ s_i &= i\mu. \end{aligned}$$

Each individual gives birth at rate λ and dies at rate μ.

Still an affine model; can have truly non-linear rates.

Overcrowding Models

A logistics type model has quadratic deaths that limit large populations

$$\begin{aligned} a_i &= i\lambda \\ s_i &= i\mu_1 + i^2\mu_2 \end{aligned}$$

for $\lambda, \mu_1, \mu_2 \geq 0$.

Average Change

Later study the martingale problem which implies average change.

If X is a birth-death process with (state-dependent rates) rates a_i and s_i, then its expected population follows

$$E\left[X_t\right] = E\left[X_0\right] + \int_0^t E\left[a_{X_s} - s_{X_s}\right] ds$$

For linear growth and immigration:

$$E\left[X_t\right] = E\left[X_0\right] + \int_0^t \left(\lambda - \mu\right) E\left[X_s\right] + \theta \, ds,$$

which has the unique solution (see problems)

$$E\left[X_t\right] = e^{(\lambda-\mu)t} E\left[X_0\right] + \frac{\theta}{\lambda - \mu}\left[e^{(\lambda-\mu)t} - 1\right]$$

Otherwise in non-linear case, the expected population equation is difficult to use because one expectation will lead to a different one.

Transitions

As for queues, the next state transition probabilities and sojourn time for state i are:

$$p\left(i \to i+1\right) = \frac{a_i}{a_i + s_i}, \quad p\left(i \to i-1\right) = \frac{s_i}{a_i + s_i},$$

$$f_{T_i}\left(t\right) = \left(a_i + s_i\right) \exp\left(-\left(a_i + s_i\right) t\right)$$

i) the probabilities are for where it will go *when* it changes and it can only go up or down by 1, and

ii) it will take a $\left(a_i + s_i\right)$-exponential time to change from state i.

Steady State

Population goes up or down by one so our queue formulae still work.

"More deaths than births" if: $1 + \frac{a_0}{s_1} + \frac{a_0 a_1}{s_1 s_2} + \cdots < \infty$, then

$$\pi(0) = \left[1 + \frac{a_0}{s_1} + \frac{a_0 a_1}{s_1 s_2} + \cdots \right]^{-1}$$

$$\pi(i) = \frac{a_{i-1} \cdots a_0}{s_i \cdots s_1} \pi(0).$$

This is what we know about total populations (in a box).

Develop tools to analyze these and other systems better.

Birth-death Approximations

Modified birth-death processes provide useful approximation if:

1) Move up and down by some small $\frac{1}{N}$ instead of simply 1. In this case, the rates a_i, s_i usually depend upon N as well.

2) Population can become negative. This is done by removing the restriction $s_0 = 0$.

Situation where 1) is true referred to as a *generalized birth-death* process. Our steady state formula may still hold with modest changes.

Model where both 1) and 2) are true called *generalized birth-death* process or a *generalized random walk* below.